Preconditioning Techniques for Simulating Floating Structures

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Outline (1)

Goal

- Develop efficient preconditioners to simulate large floating structures.

Question

- Can we reduce the compute resources needed while solving large problems?

Implementation

- Can we implement this on Gridap? (Strong-form FSI coupling)
- Can we use Trilinos as a part of the preconditioner?
- Can we build an efficient interface between Trilinos and Gridap?
- Can we solve this problem quickly?

Outline (2)

Difficulties / Problem Characteristics

- It is a multi-physics problem (Fluid and Solid).
- It is a mixed-dimensional problem (Thin-solid assumption).
- High-added mass effects.
  - Low solid density (floating structure)
  - Geometric factors - Long, thin, and slender solids.

Collaboration

- Alexander Heinlein (TU Delft)
- Oriol Colomés (TU Delft)
- Filipe Cumaru (TU Delft)

Introduction to Preconditioning

Why use a Preconditioner

- Reduce the condition number. (Easier to solve!)
- Multiphysics problems are heterogeneous.
  - Easy to solve each physics block.
  - Hard to solve the combined monolithic system.
- Introduce DD methods that scale well in parallel.

Overlapping DD methods

- Alternative Schwarz Algorithm
- One Level Additive Schwarz methods
- Two Level Additive Schwarz methods

Non-Overlapping DD methods

- Dirichlet-Neumann Algorithm
- Robin-Robin Algorithm
- Robin-Neumann Algorithm

Overlapping DD methods (1)

Alternative Schwarz methods

- One of the older DD methods introduced in 1870 by H. A. Schwarz.
- Iterative Algorithm.
- Cannot be parallelized.

$$\frac{\partial^2 u_l^{k+1}}{\partial x^2} = 1 \text{ where } x \in \Omega_l$$

$$u_l^{k+1} = u_r^k \text{ on } \Gamma_l$$

$$\frac{\partial^2 u_r^{k+1}}{\partial x^2} = 1 \text{ where } x \in \Omega_r$$

$$u_r^{k+1} = u_l^{k+1} \text{ on } \Gamma_r$$

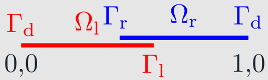


Figure: Domain Ω

Overlapping DD methods (2)

Adopted from Toselli and Widlund [3].

1-Level Additive Schwarz

- Preconditioner:

$$P_{1AS} = \sum_i (R_i^T A_i^{-1} R_i)$$

- Condition Number estimate

$$\kappa(P_{1AS}^{-1} A) \leq C(1 + \frac{1}{H\delta})$$

2-Level Additive Schwarz

- Preconditioner:

$$P_{2AS} = R_0^T A_0^{-1} R_0 + \sum_i (R_i^T A_i^{-1} R_i)$$

- Condition Number estimate

$$\kappa(P_{2AS}^{-1} A) \leq C(1 + \frac{H}{\delta})$$

Overlapping DD methods (3)

Results

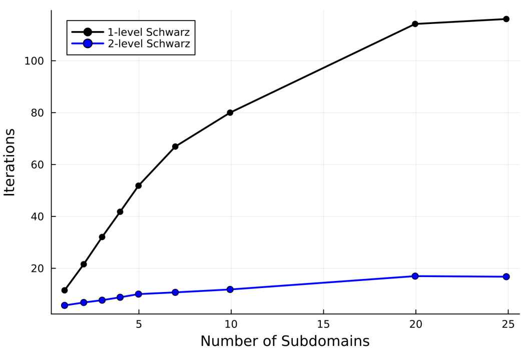


Figure: Scaling analysis

Non-Overlapping DD methods (1)

Simple Elliptic problem

$$a_1 \Delta u_l = f_1 \text{ on } \Omega_l$$

$$a_2 \Delta u_r = f_2 \text{ on } \Omega_r$$

$$u_{l\Gamma} = u_{r\Gamma} \text{ on } \partial_{lr}$$

$$a_1 \nabla u_{l\Gamma} \cdot n_l + a_2 \nabla u_{r\Gamma} \cdot n_r = 0 \text{ on } \partial_{lr}$$

$$\begin{bmatrix} A_{II} & A_{I\Gamma} & 0 & 0 \\ 0 & I & 0 & -I \\ 0 & 0 & B_{II} & B_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} & B_{\Gamma I} & B_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} u_{lI}^{n+1} \\ u_{l\Gamma}^{n+1} \\ u_{rI}^{n+1} \\ u_{r\Gamma}^{n+1} \end{bmatrix} = \begin{bmatrix} r_l \\ 0 \\ r_r \\ r_\Gamma \end{bmatrix}$$

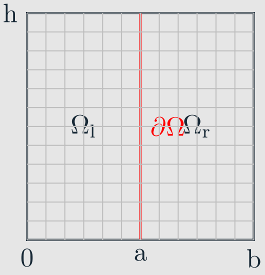


Figure: Domain Ω

Non-Overlapping DD methods (2)

Dirichlet-Neumann Method

- Algorithm:

$$a_1 \Delta u_l^{k+1} = f_1 \quad \text{on } \Omega_l$$

$$u_{l\Gamma}^{k+1} = u_{r\Gamma}^{k} \quad \text{on } \partial_{lr}$$

$$a_2 \Delta u_r^{k+1} = f_2 \quad \text{on } \Omega_r$$

$$a_1 \nabla u_{l\Gamma}^{k+1} \cdot n_l + a_2 \nabla \hat{u}_{r\Gamma}^{k+1} \cdot n_r = 0 \quad \text{on } \partial_{lr}$$

$$u_{r\Gamma}^{k+1} = \omega \hat{u}_{r\Gamma}^{k+1} + (1-\omega) u_{r\Gamma}^{k}$$

- Preconditioner:

$$P_{DN} = \begin{bmatrix} A_{II} & A_{I\Gamma} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & B_{II} & B_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} & B_{\Gamma I} & B_{\Gamma\Gamma} \end{bmatrix}$$

Non-Overlapping DD methods (3)

Adopted from Marini and Quarteroni [2].

Dirichlet-Neumann Method

- Convergence Criteria

$$\text{relative error} = 1 - \omega\left(\frac{a_1}{a_2}\frac{\tanh\left(\frac{n\pi(b-a)}{H}\right)}{\tanh\left(\frac{n\pi a}{H}\right)} + 1\right) \text{ where } n \in \mathbb{N}$$

- Optimum Relaxation Parameter

$$\omega_{opt} = \frac{1}{\left(\frac{a_1}{a_2}\frac{\tanh\left(\frac{\pi(b-a)}{b}\right)}{\tanh\left(\frac{\pi a}{b}\right)} + 1\right)}$$

Non-Overlapping DD methods (4)

Robin-Robin Method

- Algorithm:

$$a_1 \Delta u_l^{k+1} = f_1 \quad \text{on } \Omega_l$$

$$\alpha_f u_{l\Gamma}^{k+1} + a_1 \nabla u_{l\Gamma}^{k+1} \cdot n_l = \alpha_f u_{r\Gamma}^{k} - a_2 \nabla u_{r\Gamma}^{k} \cdot n_r \quad \text{on } \partial_{lr}$$

$$a_2 \Delta u_r^{k+1} = f_2 \quad \text{on } \Omega_r$$

$$\alpha_s u_{l\Gamma}^{k+1} + a_1 \nabla u_{l\Gamma}^{k+1} \cdot n_l = \alpha_s u_{r\Gamma}^{k+1} - a_2 \nabla \hat{u}_{r\Gamma}^{k+1} \cdot n_r \quad \text{on } \partial_{lr}$$

$$u_{r\Gamma}^{k+1} = \omega \hat{u}_{r\Gamma}^{k+1} + (1-\omega) u_{r\Gamma}^{k}$$

- Mapping to the original problem:

$$Q = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \alpha_f I & 0 & I \\ 0 & 0 & I & 0 \\ 0 & -\alpha_s I & 0 & I \end{bmatrix}$$

Non-Overlapping DD methods (5)

Robin-Robin Method

- Preconditioner:

$$P_{RR} = Q^{-1} \begin{bmatrix} A_{II} & A_{I\Gamma} & 0 & 0 \\ A_{\Gamma I} & \alpha_f I + A_{\Gamma\Gamma} & 0 & 0 \\ 0 & 0 & B_{II} & B_{I\Gamma} \\ A_{\Gamma I} & -\alpha_s I + A_{\Gamma\Gamma} & B_{\Gamma I} & B_{\Gamma\Gamma} + \alpha_s I \end{bmatrix}$$

Non-Overlapping DD methods (6)

Sample Results

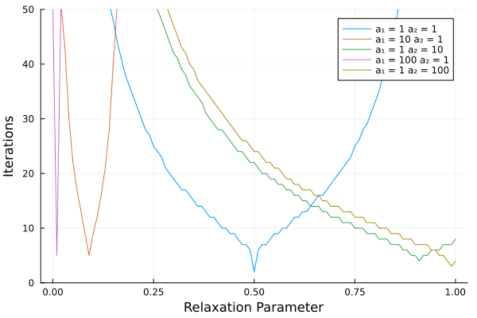


Figure: Dirichlet-Neumann

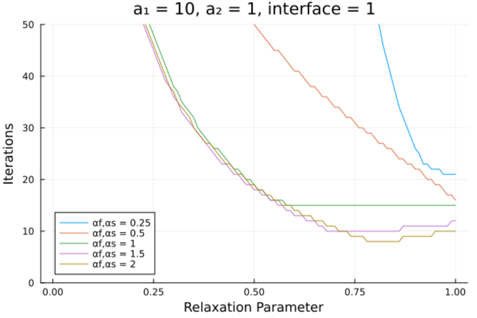


Figure: Robin-Robin

Non-Overlapping DD methods

Summary

Diffusion Contrast	$a_1 = a_2$	$a_1 = a_2$	$a_1 > a_2$	$a_1 > a_2$	$a_1 < a_2$	$a_1 < a_2$
Interface Position	Center	Asymmetric	Center	Asymmetric	Center	Asymmetric
Dirichlet-Neumann Method	Best	Best	Bad	Bad	Good	Good
Dirichlet-Robin Method	Good	Good	Good	Good	Good	Good
Robin-Neumann Method	Good	Good	Bad	Bad	Good	Good
Robin-Robin Method	Good	Good	Best	Best	Best	Best

Table: Summary for Non-Overlapping methods applied on the coupled elliptic PDE

Toy FSI Problem (1)

Adopted from Fernández, Mullaert, and Vidrascu [1].

Computational Domain

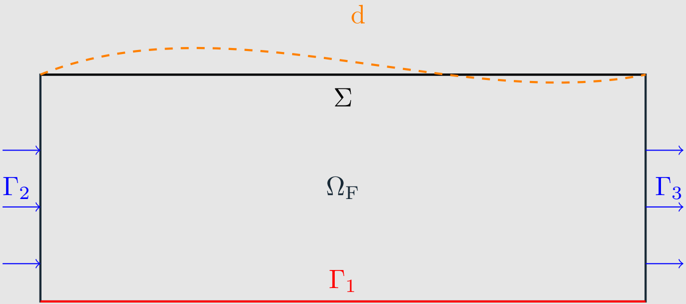


Figure: Computational domain for the toy FSI problem

Toy FSI Problem (2)

Governing Equations

- Fluid Domain

$$\rho_f \frac{\partial u}{\partial t} + \nabla p = 0 \text{ on } \Omega_f$$

$$\nabla \cdot u = 0 \text{ on } \Omega_f$$

$$u \cdot n_\Gamma = 0 \text{ on } \Gamma_1$$

$$p = 0 \text{ on } \Gamma_2$$

- Solid Domain and Interface

$$\rho_s \epsilon \ddot{d} + Ld = p \text{ on } \Sigma$$

$$u = \dot{d} \text{ on } \Sigma$$

Toy FSI Problem (3)

Discrete Problem

- Fluid Problem

$$\begin{bmatrix} C_{II} & G_I & C_{I\Gamma} \\ C_{\Gamma I} & G_\Gamma & C_{\Gamma\Gamma} \\ D_I & D_\Gamma & 0 \end{bmatrix} \begin{bmatrix} uf_I \\ uf_\Gamma \\ p \end{bmatrix} = \begin{bmatrix} rf_I \\ rf_\Gamma \\ rf_p \end{bmatrix}$$

- Solid Problem

$$[N_{\Gamma\Gamma}][ds] = [rs]$$

- Coupled Problem

$$\begin{bmatrix} C_{II} & G_I & C_{I\Gamma} & 0 \\ D_I & 0 & D_\Gamma & 0 \\ 0 & 0 & I & -\partial_t(I) \\ C_{\Gamma I} & G_\Gamma & C_{\Gamma\Gamma} & N_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} uf_I \\ p \\ uf_\Gamma \\ ds \end{bmatrix} = \begin{bmatrix} rf_I \\ rf_p \\ 0 \\ rf_\Gamma + rs \end{bmatrix}$$

Toy FSI Problem (4)

Discrete Problem

- Coupled Problem (Robin-Robin Transmission Conditions)

$$\begin{bmatrix} C_{II} & G_I & C_{I\Gamma} & 0 \\ D_I & 0 & D_\Gamma & 0 \\ C_{\Gamma I} & G_\Gamma & \alpha_f I + C_{\Gamma\Gamma} & -\alpha_f * \partial_t(I) + N_{\Gamma\Gamma} \\ C_{\Gamma I} & G_\Gamma & -\alpha_s I + C_{\Gamma\Gamma} & \alpha_s * \partial_t(I) + N_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} uf_I \\ p \\ uf_\Gamma \\ ds \end{bmatrix} = \begin{bmatrix} rf_I \\ rf_p \\ rf_\Gamma + rs \\ rf_\Gamma + rs \end{bmatrix}$$

Preconditioners for the FSI Problem (1)

Dirichlet-Neumann

$$P_{DN} = \begin{bmatrix} C_{II} & G_I & C_{I\Gamma} & 0 \\ D_I & 0 & D_\Gamma & 0 \\ 0 & 0 & I & 0 \\ C_{\Gamma I} & G_\Gamma & C_{\Gamma\Gamma} & N_{\Gamma\Gamma} \end{bmatrix}$$

Robin-Robin

$$P_{RR} = Q^{-1} * \begin{bmatrix} C_{II} & G_I & C_{I\Gamma} & 0 \\ D_I & 0 & D_\Gamma & 0 \\ C_{\Gamma I} & G_\Gamma & \alpha_f I + C_{\Gamma\Gamma} & 0 \\ C_{\Gamma I} & G_\Gamma & -\alpha_s I + C_{\Gamma\Gamma} & \alpha_s * \partial_t(I) + N_{\Gamma\Gamma} \end{bmatrix}$$

Preconditioners for the FSI Problem (2)

Robin-Neumann

$$P_{RN} = Q^{-1} * \begin{bmatrix} C_{II} & G_I & C_{I\Gamma} & 0 \\ D_I & 0 & D_\Gamma & 0 \\ C_{\Gamma I} & G_\Gamma & \alpha_f I + C_{\Gamma\Gamma} & 0 \\ C_{\Gamma I} & G_\Gamma & C_{\Gamma\Gamma} & N_{\Gamma\Gamma} \end{bmatrix}$$

Added Mass Operator

Let the operator $M_A : H^{1/2}(\Sigma) -> H^{1/2}(\Sigma)$ be defined for each $g \epsilon H^{-1/2}(\Sigma)$, where we set $M_A(g) = q|_\Sigma$ where $q \epsilon H^1(\Omega)$ for the following problem.

$$-\Delta q = 0 \text{ on } \Omega$$

$$\frac{\partial q}{\partial n} = g \text{ on } \Sigma$$

FSI Preconditioners Stability (1)

Modified Problem

- Applying the divergence operator on the original problem.

$$-\Delta \mathrm{p} = 0 \text{ on } \Omega_{\mathrm{f}}$$

$$\nabla \mathrm{p} \cdot \mathrm{n}_{\mathrm{fs}} = 0 \text{ on } \Gamma_1$$

$$\rho_{\mathrm{s}} \epsilon \ddot{\mathrm{d}} + \mathrm{Ld} = \mathrm{p} \cdot \mathrm{n}_{\mathrm{fs}} \text{ on } \Sigma$$

$$\nabla \mathrm{p} \cdot \mathrm{n}_{\mathrm{fs}} = -\rho_{\mathrm{f}} \ddot{\mathrm{d}} \text{ on } \Sigma$$

Interface Equation (Dirichlet-Neumann method)

$$\rho_{\mathrm{s}} \epsilon [\frac{\hat{\mathrm{d}}_{\mathrm{k+1}} - 2\mathrm{d}^{\mathrm{n}} + \mathrm{d}^{\mathrm{n-1}}}{\delta \mathrm{t}^2}] + \rho_{\mathrm{f}} \mathrm{M}_{\mathrm{A}} [\frac{\mathrm{d}_{\mathrm{k}} - 2\mathrm{d}^{\mathrm{n}} + \mathrm{d}^{\mathrm{n-1}}}{\delta \mathrm{t}^2}] + \mathrm{L} \hat{\mathrm{d}}_{\mathrm{k+1}} = 0 \text{ on } \Sigma$$

FSI Preconditioners Stability (2)

Dirichlet-Neumann method

- Convergence Criteria:

$$|\frac{(1-\omega)(\rho_s\epsilon + L\delta t^2) - \omega\rho_f\mu_i}{\rho_s\epsilon + L\delta t^2}| < 1$$

- Range of acceptable Relaxation Parameter:

$$0 < \omega < \frac{2(\rho_s\epsilon + L\delta t^2)}{\rho_s\epsilon + \rho_f\mu_{max} + L\delta t^2}$$

Robin - Neumann method

- Optimized Relaxation parameter (Fernández et al. (2013))

$$\alpha_f = \frac{\rho_s\epsilon}{\Delta t} + L\Delta t$$

FSI Preconditioners Stability (3)

Robin-Neumann Method

- Interface Equation:

$$\left(\frac{\mathrm{L}\Delta \mathrm{t}^2 + \rho_{\mathrm{s}}\epsilon}{\rho_{\mathrm{f}}}\mathrm{M}_{\mathrm{A}}^{-1} + \mathrm{I}\right)\mathrm{p}^{\mathrm{k}+1} = \mathrm{f}(\mathrm{d}^{\mathrm{n}})$$

- Above Difference equation is an explicit update equation.
- This method is not affected by the Added mass effect as the above difference equation is always stable.

FSI Preconditioners Stability (4)

Results

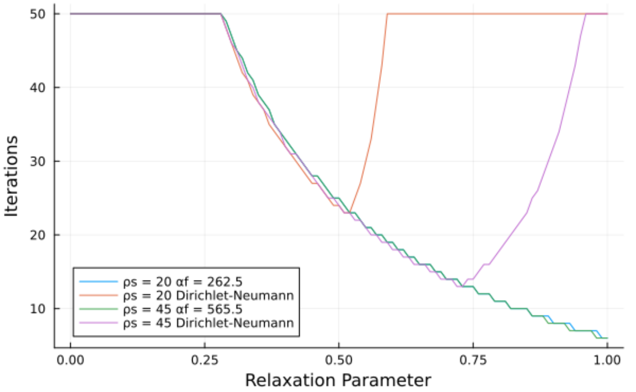


Figure: FSI Preconditioner Stability

Parallel Preconditioners for the FSI problem (1)

Linear System

$$\begin{bmatrix} N_{\Gamma\Gamma} & C_{\Gamma I} & C_{\Gamma\Gamma} & G_{\Gamma} \\ 0 & C_{II} & C_{I\Gamma} & G_{I} \\ -\partial_t(I) & 0 & I & 0 \\ 0 & D_I & D_\Gamma & 0 \end{bmatrix} \begin{bmatrix} d \\ u_i \\ u_\Gamma \\ p \end{bmatrix} = \begin{bmatrix} r_{f\Gamma} + r_s \\ r_f \\ 0 \\ r_p \end{bmatrix}$$

Dirichlet-Neumann Preconditioner

$$P_{DN_{parallel}} = \begin{bmatrix} N_{\Gamma\Gamma} & C_{\Gamma I} & C_{\Gamma\Gamma} & G_{\Gamma} \\ 0 & C_{II} & C_{I\Gamma} & G_{I} \\ 0 & 0 & I & 0 \\ 0 & D_I & D_\Gamma & 0 \end{bmatrix}$$

Parallel Preconditioners for the FSI problem (2)

Mapping matrix

$$Q = \begin{bmatrix} I & 0 & -\alpha_s I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & \alpha_f I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

Robin-Robin Preconditioner

$$P_{RR_{parallel}} = \begin{bmatrix} \alpha_s * \partial_t(I) + N_{\Gamma\Gamma} & C_{\Gamma I} & -\alpha_s I + C_{\Gamma\Gamma} & G_\Gamma \\ 0 & C_{II} & C_{I\Gamma} & G_I \\ 0 & C_{\Gamma I} & \alpha_f I + C_{\Gamma\Gamma} & G_\Gamma \\ 0 & D_I & D_\Gamma & 0 \end{bmatrix} * Q^{-1}$$

Parallel Preconditioners for the FSI problem (3)

Generic Preconditioner structure

$$P = \begin{bmatrix} \text{Solid Block} & \text{Coupling Terms} \\ 0 & \text{Fluid Block} \end{bmatrix}$$

Solvers Implementation

- Block preconditioner applied with a FGMRES solver implemented using the Gridap solvers package.
- Overlapping Schwarz preconditioners applied on a GMRES solver to solve the blocks within the preconditioner implemented using Trilinos.

FSI Block Preconditioners Results (1)

Optimal Robin Parameter

$$\alpha_{\mathrm{f}} = \gamma(\frac{\rho_{\mathrm{s}}\epsilon}{\Delta \mathrm{t}} + \mathrm{L}\Delta \mathrm{t})$$

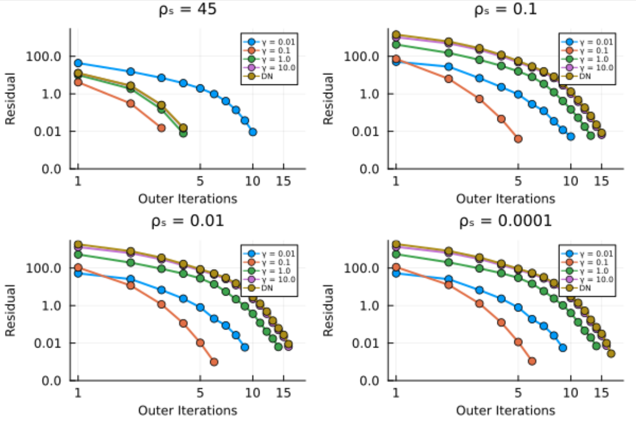
Variation of Length

	Length = 6	Length = 50	Length = 100	Length = 1000
Dirichlet-Neumann	17	38	52	157
Robin-Neumann	6	6	6	6

Table: Number of outer iterations for FSI block preconditioners

FSI Block Preconditioners Results (2)

Solid Density Variation



HPC Implementation

Implementation

- Developed New functionalities in Gridap to implement strong form coupling in serial and parallel.
- Developed an interface between Trilinos and Gridap.

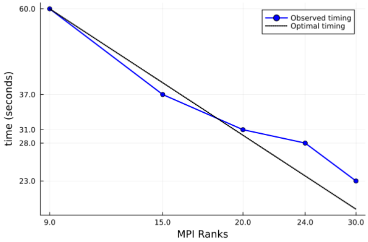


Figure: Strong scaling analysis

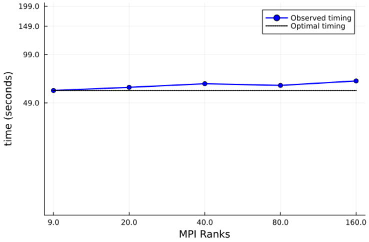


Figure: Weak scaling analysis

Conclusions

- The Robin-Neumann Preconditioner appears to be the robust block preconditioner.
  - It scales with length of the domain.
  - It scales with the Solid density of the domain.
- The implementation of the system on Gridap and Trilinos scales and works well.
  - Demonstrated the ability to solve large problems on DelftBlue.

Discussions and further work

- Better understand the convergence behavior of the Schwarz method for the Fluid and Solid block.
- Conduct a parametric analysis and implement the block preconditioners for the floating structure problem.
- Next Phase : Extend the current framework for non-linear problems.
- Other implementations - Solve the preconditioner only once and apply as a linear operator.
- Limitation - Current implementation only works for conforming meshes. Explore approaches based on weak coupling and Lagrange multipliers.

Bibliography

[1] Miguel A. Fernández, Jimmy Mullaert, and Marina Vidrascu. "Explicit RobinNeumann schemes for the coupling of incompressible fluids with thin-walled structures". In: Computer Methods in Applied Mechanics and Engineering 267 (Oct. 2013), pp. 566–593. doi: 10.1016/j.cma.2013.09.020. url: https://doi.org/10.1016/j.cma.2013.09.020.

[2] L. D. Marini and A. Quarteroni. "A relaxation procedure for domain decomposition methods using finite elements". In: Numerische Mathematik 55.5 (Sept. 1989), pp. 575–598. doi: 10.1007/bf01398917. url: https://doi.org/10.1007/bf01398917.

[3] Andrea Toselli and Olof Widlund. Domain Decomposition Methods - Algorithms and Theory. Springer Science Business Media, June 2006. url: http://books.google.ie/books?id=h7EVoI2g1nkC&printsec=frontcover&dq=Domain+Decomposition+Methods:+Algorithms+and+Theory&hl=&cd=1&source=gbs_api.

Thank you for your attention!

Questions ?